

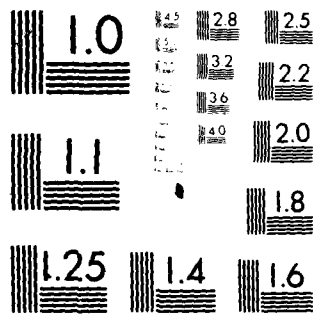
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POSSIBILITIES VERSUS FUZZY PROBABILITIES--  
TWO ALTERNATIVE DECISION AIDS

Anthony N. S. Freeling

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November 1981

Prepared for:

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Naval Analysis Programs  
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POSSIBILITIES VERSUS FUZZY PROBABILITIES--TWO ALTERNATIVE DECISION AIDS

By

Anthony N.S. Freeling

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## SUMMARY

In this paper we look at the two major approaches that have been proposed using fuzzy set theory to help in the analysis of decision trees, one based on possibility theory, and one on a fuzzy extension of probability theory. We conclude that the first approach requires an operational definition of possibility in order to be sufficiently compelling to justify its use in preference to standard decision analysis. We suggest that this work should prompt us to consider the potential of using different connectives in different decision contexts. It is stressed that the second approach is an extension of the standard paradigm, rather than an alternative, as has sometimes been assumed, and we suggest that it is more promising than the first approach. We show how the extended theory may be used to help understand and deal with inconsistent probability (or utility) assessments, and we introduce the concept of the "value of perfect coherence" as an aid to calculate the value of a decision analysis.

#### ACKNOWLEDGMENTS

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## TABLE OF CONTENTS

1.0 INTRODUCTION-----	1
2.0 POSSIBILISTIC DECISION-MAKING-----	3
3.0 FUZZY PROBABILISTIC DECISION-MAKING-----	11
4.0 INCONSISTENCY AND INCOHERENCE-----	19
5.0 SUMMARY AND CONCLUSIONS-----	23
REFERENCES-----	25

## 1.0 INTRODUCTION

Decision analysis (DA) is a technique which can be used to help a decision maker (DM) who is faced with a problem involving uncertainty and/or utility considerations. DA has been well formulated for more than a decade [23], [17], and successful consulting firms have grown up whose expertise lies almost entirely within the field of decision analysis. Several successful applications of the paradigm have been reported, e.g. [18], [19], and [13]. The technique as applied rests on the assumptions that a decision tree can be drawn summarizing the situation and that all requisite probabilities and utilities may be assessed from the decision-maker.

These assumptions are not, however, always valid, and so various modifications and extensions of the paradigm have been proposed. In this paper we look at two methods that have been proposed which draw on the theory of fuzzy sets [31], [32], which was developed as a means for modeling imprecision. Each method accepts the structuring assumption, but tries to relax the necessity for precise numbers that the basic paradigm uses to represent the DM's uncertainty and values. A relaxation of that requirement is worth seeking, since a full elicitation of probabilities and utilities is always time-consuming, and sometimes extremely difficult. Furthermore, DM's often feel that they would prefer not to have to give numerical values to the various elements of the decision tree. In this paper we shall focus our attention on the modelling of uncertainty, for ease of exposition.

The first approach we discuss is represented by Yager, [30], and also by Whalen, [28]. They attempt to replace the need for measuring values on a cardinal scale, as is the case with probabilities and utilities, with a requirement for measuring values on an ordinal scale (i.e. where knowledge of ordering of values is all that is required) by using the concepts of possibility theory [32]. We look at this approach in Section 2.0

The second approach at which we shall look has been studied by Watson, Weiss and Donnell [27], Freeling [7], [8] and by Adamo [1]. Here the principle of maximizing expected utility is retained, but the input probabilities and utilities are allowed to be fuzzy, rather than precise. They may be assessed only linguistically (e.g. we may know only that event A is "quite probable"), and using the extended arithmetic operations, as described for example in [4], "fuzzy expected utilities" are calculated and then compared. In Section 3.0 we look at the potential and justification for this approach.

In Section 4.0 we show how postulating fuzzy probabilities may be useful in explaining and coping with inconsistencies in a DM's probability assessments, a problem that often occurs in applied decision analysis, and one that has begun to receive some attention in the literature [21], [9]. In Section 5.0 we make some concluding remarks and suggestions for further research in this area. This paper contains very little detailed mathematics, but rather we attempt to explain the value and potential of the approaches discussed. For more detailed expositions of the methods, the reader should turn to the references mentioned above. In particular, we do not devote space to defining the basic concepts of fuzzy sets, which are well treated elsewhere.

## 2.0 POSSIBILISTIC DECISION-MAKING

To illustrate the approach taken in [30] and [28], we shall use an example from [30] (see Figure 1). This will be recognized as a standard, and very simple, decision tree. The (somewhat sexist) problem it describes concerns the dilemma of a young man (Bill) at a party, who is very attracted to a certain young lady there. He feels that he would have more success (left inexplicit in meaning!) if he waited until later in the party to approach her. However, he observes Stanley, who has a reputation as a ladies' man, at the party, and he feels that if Stanley approaches his target first she will succumb to Stanley's charms. Thus our hero faces the decision of whether to approach now, with less likelihood of success, or to wait until the appropriate moment, and hope that Stanley approaches someone else.

Using the usual DA methodology, Bill would assess the probabilities at nodes 1, 2 and 3, and utilities at each of the end points. Then if  $p_1$  is the probability of success at the right time,  $p_2$  the probability of immediate success, and  $p_3$  the probability that Stanley approaches someone else; and if  $u_1$  is the utility attached to endpoint 1, the expected utility of waiting is

$$(1-p_3)u_1 + p_3(p_1u_2 + (1-p_1)u_3)$$

and the expected utility of an immediate approach is

$$p_2u_4 + (1-p_2)u_5$$

and Bill simply has to choose that option having the highest expected utility.

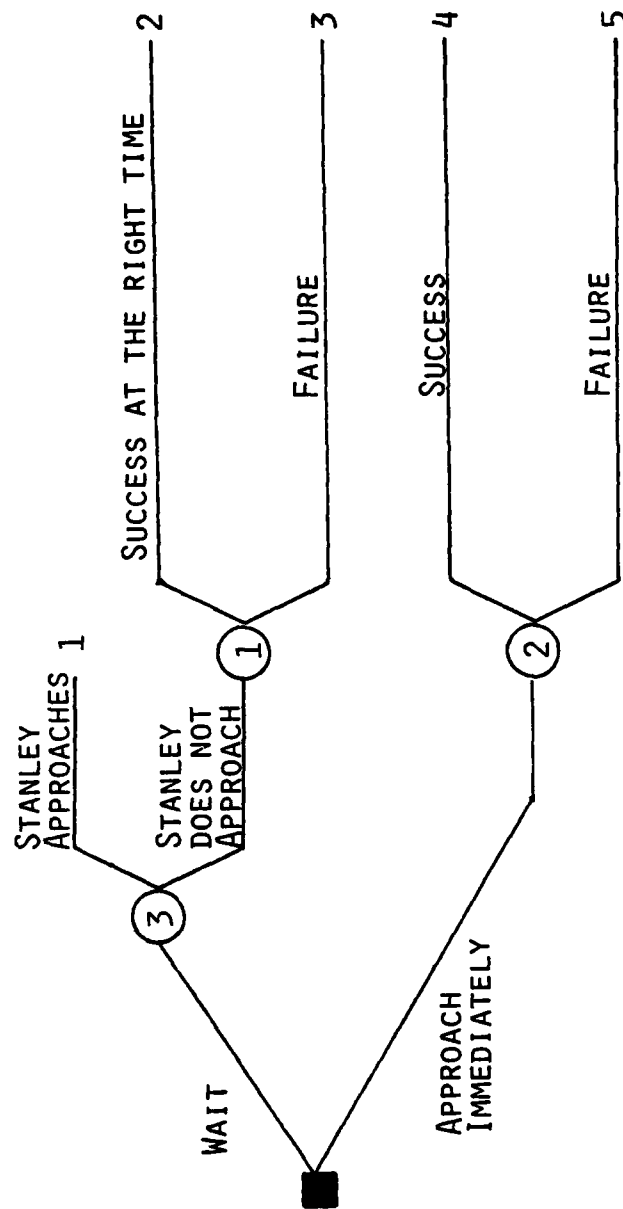


FIGURE 1

This is of course an example of the general principle of maximizing expected utility, where if the probability of reaching endpoint  $i$  is  $p_i$  then the expected utility is

$$\sum_i p_i u_i.$$

Yager argues that the analogous quantity to maximize if we assume that the  $p_i$  are "possibilities", and the  $u_i$  the degree of membership of outcome  $i$  in the set of "good things", should be

$$\max_i (\min(p_i, u_i)).$$

This is consistent with the general tenet that in going from probability theory to possibility theory, addition maps onto maximum, and multiplication onto minimum. This arises directly from the difference in the rules for union and intersection in the two theories: whereas in probability theory the probability of an endpoint is calculated by multiplying all the probabilities of the branches reaching it, when using possibilities we must take the minimum of all the possibilities leading to that final outcome. It is now clear why only ordinal information is considered necessary with possibilistic decision-making, because the operations we use, max and min, require knowledge only of order.

In the example of Figure 1 let us assume that the utility structure is such that  $u_i$  is equal to one if and only if Bill is successful; else  $u_i$  is zero, and that this is true whether we are using probabilistic or possibilistic rules. Then using probabilities, Bill should approach only if

$$p_1 p_3 < p_2,$$

whereas using possibilities, the criterion is

$$\text{Min}(p_3, p_1) < p_2.$$

Thus far, Yager's approach may appear promising. If we can reach a decision using only ordinal information, why not do so? The difficulty lies in finding a justification for the use of the maximum and minimum operators.

Since Zadeh first proposed fuzzy set theory [31] research has been conducted to justify the use of the max-min connectives. There has been some success at developing axiomatic systems [2], [14] but psychological work attempting to discover if the connectives are descriptive of human cognitive processes has been rather negative [26], [16], [30]. It seems clear that the choice of connectives should be dependent upon the situation in which the theory is going to be used, as is argued by Gaines [15].

Probabilists have protested that they see no reason why numbers in the range  $[0,1]$  that we elicit from a DM should not be probabilities, and operated with as such. Dennis Lindley (personal communication) has expressed this perhaps the most persuasively. He suggests that if, associated with an event  $E$ , there is a number between zero and one which measures a feeling about the truth of  $E$ , then there are two questions we need to ask about such numbers:

- a) how do they combine, and
- b) what is an operational interpretation of them?

Lindley then points out that, if  $x$  is the "truth value," if we are served a penalty  $(x - 1)^2$  if  $E$  turns out to be true, and  $x^2$  if  $E$  turns out to be false, then unless the numbers  $x_i$  for different events  $E_i$  satisfy the laws of

probability, our total penalty can be reduced whatever the truth or falsity of the various  $E_i$ . Thus by using other rules for combination, we are committing ourselves to losing money for sure. This penalty scheme is an example of a scoring rule [18], the idea being that we do best if we give a value of one to  $E$  when it is true, and zero when it is false. The argument may be extended to a very wide variety of scoring rules. The operational definition of probability arises out of the concept of indifference between gambles on differing events. A further argument for the standard DA paradigm uses the gambling interpretation--if one does not bet according to the laws of probability one is open to losing money for sure, whatever the outcome of the uncertain events. This would be an example of a Dutch Book [20]. We thus see that using the max-min connectives could get very expensive!

Let us now examine the response to these points that could be made in defense of possibilistic decision-making. First, we might argue, following Whalen [28], that we do not need to use a number in  $[0,1]$  to represent the degree of truth of  $E$ , but simply an ordered, qualitative scale, upon which max-min operations are well-defined. This however does not avoid the problem since we are still required to make comparisons such as

"Is it more possible that  $E$  is true than it is true that outcome  $A$  is regrettable?"

It is not at all clear whether such comparisons can be made with any more ease than assessing numbers, and indeed, subjects may even use an implicit number scale to effect the comparison. In any case, the qualitative scale could be mapped onto  $[0,1]$  and Lindley's arguments would still hold.



A second obvious response to Lindley's arguments is that there is no reason why Bill should accept the reward structure implied by a scoring rule. This response is further strengthened by research indicating that using scoring rules to motivate subjects to provide good probability assessments, has not been an unqualified success (see for example [22]). Our understanding of why this is so, and why Lindley's argument is not fully persuasive, lies in a distinction between the normative and the prescriptive aspects of the DA paradigm. This distinction is discussed more fully in a recent report [12]: to summarize, a normative theory discusses how a perfectly rational person would act; a prescriptive theory how a real person should act. The two concepts are distinct, though often confused.

The arguments put forward by Lindley are most persuasive normatively: if a perfectly rational being acted in accordance with the scoring rule, he would provide numbers which satisfied the probability axioms and which could be interpreted in the operational sense given above. A real DM, on the other hand, will provide numbers which satisfy the axioms only approximately, and thus the DA paradigm is only partially satisfactory as a prescription of how to use these numbers. Yager indeed has developed the ideas in [30] as a means for coping with the limited ability of a DM to provide probabilities. However, in order to justify the use of the fuzzy calculus, a new operational interpretation is required. When eliciting probabilities, the gambling interpretation helps the DM in his assessment. If max-min possibilities are required, a relevant interpretation will be required for the DM. Yager has not provided this. (In a recent report [12] we suggest an interpretation in terms of weights of evidence, but this seems appropriate only for inference,

rather than decision problems.) Without any such interpretation, we cannot expect the DM to provide values which should be used as inputs to Yager's method. Indeed, in many situations a DM might be expected to have a concept of chance, in which case elicited values might approximate probabilities, and using max-min, erroneous conclusions result. For example, in Figure 1, suppose  $p_1 = 0.7$ ,  $p_2 = 0.4$ , and  $p_3 \approx 0.5$ , then using max-min, Bill should choose to wait; using probability he would approach immediately. If Bill had been thinking of probabilities, using max-min will have led to the wrong decision. In the absence, then, of a normative framework which leads to the max-min connectives, and which further provides an operational definition of the numbers used, the modification of the DA paradigm suggested in [30] and [28] must be considered unsatisfactory as a decision aid.

We do not however feel that the significance of the work in these two papers should be overlooked. If one accepts the "bounded rationality" of a real DM, then Lindley's arguments for the use of multiplication and addition to deal with probabilities lose their force. It may be that neither of the two types of operations so far discussed are best for decision aiding, but that we ought to investigate the use of connectives which lie somewhere between the probability ones and the possibility ones. Maybe a different operator should be used in different cases, drawing perhaps from the infinite family of connectives suggested by Yager himself, in another paper [29]. Thus rather than stating, as with the DA paradigm, that

"to calculate the value of a decision option we must multiply together the probabilities leading to each endpoint and the utility at that endpoint, and then sum over all the possible endpoints,"

or as in [30], that

"to calculate the value of a decision option we must take the minimum of all the possibilities leading to an endpoint and of the goodness of that endpoint, and then take the maximum over all endpoints,"

our generalized paradigm would state that

"IF the numbers elicited from the DM are such that operation \* is appropriate for intersection, and operation @ is appropriate for union, THEN to calculate the value of a decision option we must perform \* on the numbers associated with the uncertainties leading to each endpoint and on the value of that endpoint, and then perform @ over all the possible endpoints."

This very complicated expression greatly increases the flexibility of the decision-aid, but maybe at the cost of making it intractable, since we would need also, for each DM, to assess his \* and @ functions. However the techniques of psychological scaling may prove applicable here, and the approach is probably worth looking into further. Herein, then, lies the major value of [30] and [28]--they have prompted us to look again at the standard DA paradigm and see alternatives to it that were not at all apparent until the new concepts were produced.

### 3.0 FUZZY PROBABILISTIC DECISION-MAKING

The use of fuzzy sets made in [27], [7], [8] and [1] is very different from that of [30] and [28]. Whereas in [30] and [28], the authors propose what is essentially an alternative to probability theory, the authors of [27], [7], [8] and [1] view fuzzy set theory as a means of extending probability theory. Thus, in these latter papers the authors keep the idea that one should calculate expected utility, but propose that one should use fuzzy numbers rather than crisp ones to perform this calculation. They draw on the so-called Extension Theorem, to "fuzzify" the structure of the usual DA paradigm. Instead of using the expected utility as equalling

$$\sum_i p_i \cdot u_i$$

they compute the fuzzy expected utility as

$$p_1 \odot u_1 \oplus p_2 \odot u_2 \oplus \dots \oplus p_n \odot u_n \quad ,$$

where  $\oplus$  and  $\odot$  are the binary operations of extended addition and extended multiplication, respectively. For further details concerning these extended operations, see [4], and each of the references [27], [7], [8] and [1] gives examples of their application.

The essential feature of this approach is the assumption that the probabilities and utilities assessed need not be precise. It is a matter of common observation in applied decision analysis that DM's often feel uneasy at being tied down to exact numbers, and that they will often give values which are integral multiples of 0.1. They would feel easier were they permitted to say the probability of an event was "about 0.3" than that it is 0.3. Further,

some probabilities are far more certain than others: e.g., I am far more confident that the probability of a coin falling down heads is a half, if I have observed 1000 trials, of which a half were heads, than I feel about the probability of the President of the U.S. in 2100 being a Republican. With ordinary DA I might not be able to express the difference, whereas with fuzzy sets, this can be done easily. Consider Figure 2. The graph labelled (a) depicts a probability of an event as usually assessed, but in the format of the extended theory. The probability of E is taken to be 0.5 with possibility 1, and there is no possibility of it being any other value. The situation in (b) may be more realistic--the probability may be anywhere in the range [0.3,0.7], but the possibility decreases the further one looks from 0.5. This might be an appropriate representation of the value "approximately one half." This, then, is the motivation for the approach--if we can perform DA using only approximate responses, we shall have increased the applicability of the methodology.

We must, however, examine the approach in the light of the same criticisms discussed in the previous section. The probabilists ask,

"Why are the graphs in Figure 2 not probabilistic; i.e., why do we not say the probability that P is 0.4, is a half, rather than saying the possibility is one half?"

It should be noted that Lindley's arguments are not applicable here, for it is meaningless to talk about rewards if the probability in fact turns out to be 0.4--since we are talking about subjective probabilities such an event is unverifiable. The author would argue further, in fact, that there is no such thing as "the subjective probability." Rather a DM will have, at best a

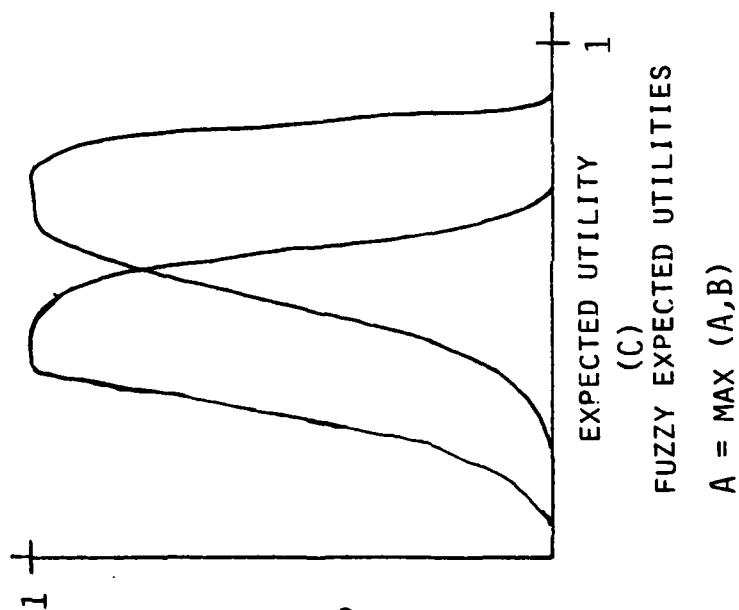
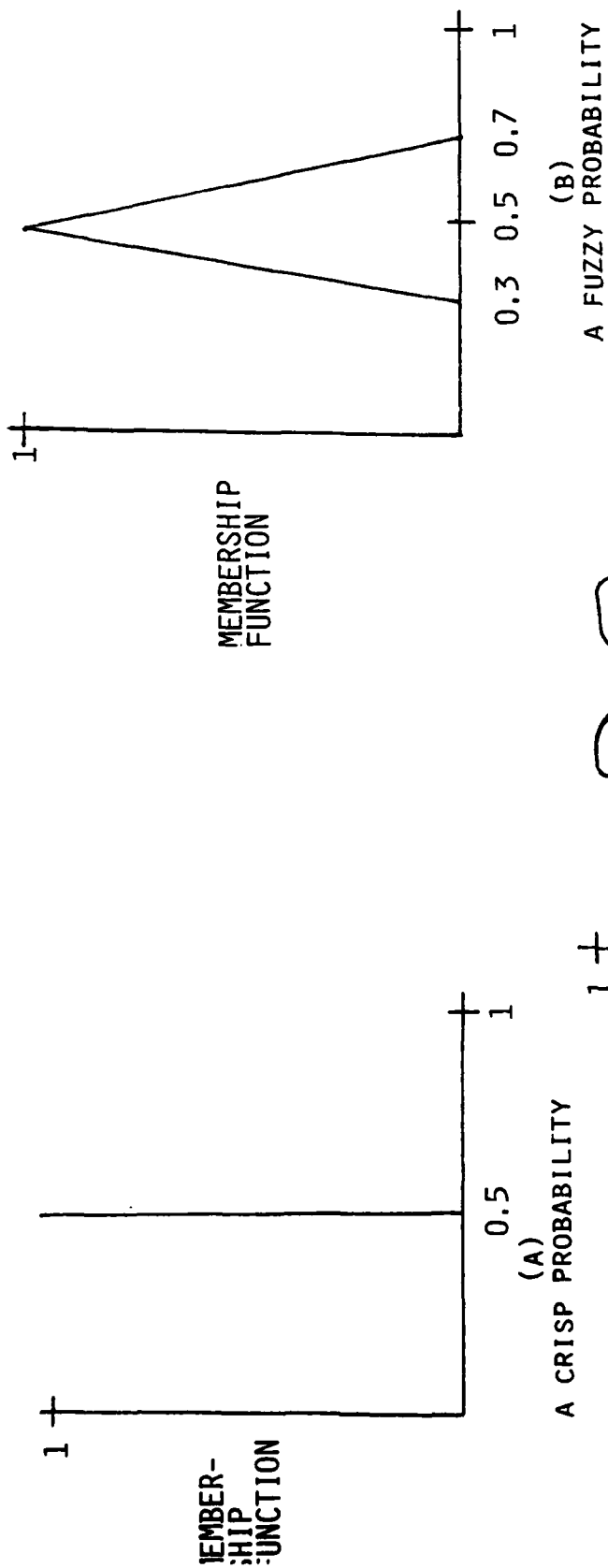


FIGURE 2

somewhat fuzzy concept of the uncertainty, and this can only be described by using fuzzy numbers, rather than the ordinary ones used in a conventional DA.

There remains, however, the very pertinent question of WHY we should operate with these "fuzzy" numbers according to the max-min operators. There are very strong reasons for wishing the max-min operators to apply; namely that

- a) the computation of the fuzzy expected utilities is very simple,
- b) the use of them allows the full machinery of fuzzy sets to be applied to the problem,
- c) there is a very appealing interpretation of the fuzzy DA as a multiple-level sensitivity analysis.

The issue becomes, "can we elicit from the DM values for the possibilities that do satisfy the fuzzy axioms?" Freeling in [7] and [8] has made an initial attempt at providing an axiomatic basis, where the "possibilities" are to be interpreted as degrees of confidence; i.e., we ask the DM how confident he is that a certain value could be the probability, the answers ranging from certain it is not, to certain that it could be. He shows how such responses could indeed be operated with under the fuzzy calculus, providing the problem is structured in a certain way. This approach needs further development, but the work so far indicates the potential for success. It will be realized that in this context, an operational definition of the membership functions has been provided--they are degrees of confidence.

One fact that should be noted before proceeding further is that the methodology discussed in this section in fact represents a double leap from the usual DA paradigm--the first extension would be to use ordinary set

theory, and allow the probability to range within an interval; i.e., the membership function would always be zero or one. This approach of defining upper and lower probabilities was in fact taken as early as the 1960's [3], [25]. In that special case of the fuzzy theory, there need be little controversy over the use of operators, for all fuzzy logics will agree with ordinary logics for grades of membership zero and one.

A second point to note is that, although the membership functions of Figure 2 are evaluated over the interval  $[0,1]$ , if one uses the axiomatic basis suggested in [8], they are only ordinal functions. For example no attempt is made to scale "very confident" compared to "quite confident." We need only know that one implies more confidence than the other. Thus the criticism that "fuzzy sets are not really fuzzy, because the membership functions are exact", is defused since with this elicitation procedure we only require the DM to distinguish between eleven levels (or less), rather than between any two real numbers.

In order to understand how a fuzzy decision analysis may be interpreted as a multi-level sensitivity analysis, let us define the level set at level  $c$  as the set of  $x$  such that  $\mu(x) > c$ . We show in [8] that if the level sets of level  $c$  in the input functions are intervals (as in Figure 2), then the level set at level  $c$  in the fuzzy expected utility is an interval. The endpoints of this interval are defined by the extremes of the level sets at level  $c$  of the input functions. So if we interpret the level sets of the inputs as defining the range within which the input functions lie, at degree of confidence  $c$ , the



range for the expected utility, at that level of confidence, is simply the level set at level  $c$  in the output. This justifies the claim that the fuzzy DA may be interpreted as a sensitivity analysis conducted at each level for which assessments were made.

That we are conducting such a multi-level sensitivity analysis is probably, in itself, sufficient motivation for pursuing this approach. This is further strengthened when it is considered that both Savage [24] and de Finetti [6] deny the legitimacy of performing sensitivity analyses, on the grounds that the subjectivist theory "proves" that the inputs to a DA are exact, and therefore that the sensitivity analysis is meaningless. Using our extended theory, where we assume that the input functions are not exact, the value of the sensitivity analysis becomes clear, and this conforms more with our intuitions of the way a DA ought to be performed. Once again we are making the distinction between the normative theory, which postulates a super-being, and a prescriptive theory which prescribes the way that a real human should act. This whole suite of ideas is explored in greater detail, and with further reference to many alternative theories of belief, in [12].

While this argument for the use of a fuzzy DA as a sensitivity analysis is compelling, it is not wholly satisfactory. Even though a DM may wish to provide only fuzzy inputs, it is typically necessary to make a non-fuzzy decision. Thus knowing that one "sort of ought to do A" is insufficient. The problem is that whereas expected utilities can ordinarily be compared directly, fuzzy expected utilities will typically be as in Figure 2(c), where

the membership functions overlap. It can then not be categorically stated that one is greater than the other. The question of how to compare the output functions is discussed at length in [8] and also in [5]. Here we shall look only at the (common) case, where the functions are such that the "extended maximum" of the two functions is equal to one of the two functions. The extended maximum is denoted  $\tilde{\text{max}}$ , and is defined by

$$\mu_{\tilde{\text{Max}}(A_1, A_2)}(y) = \sup_{\max(x_{A_1}, x_{A_2})=y} (\min(\mu_{A_1}(x_{A_1}), \mu_{A_2}(x_{A_2})))$$

This is the fuzzy extension of the maximum of two numbers. It is not always the case that  $\tilde{\text{Max}}(A_1, A_2)$  is either  $A_1$  or  $A_2$ , but it can be shown that if the two membership functions overlap just once, after one has reached its peak and before the other one has (as in Figure 2(c)), then this is true. Freeling [8] suggests that in such situations, the fuzzy maximum should be used to make the comparison. The extended maximum induces a partial order,  $>$ , on the fuzzy numbers, by defining

$$a > b \text{ if and only if } \tilde{\text{max}}(a, b) = a.$$

This partial order can be considered, where we are dealing with fuzzy expected utilities, to be an expression of "is preferred to," which, in accordance with our intuition, becomes a fuzzy concept, rather than the crisp one of Savage [24]. If two utilities are not comparable under  $>$ , this is an indication that the amount of "fuzz" in the inputs needs to be reduced.

With the use then of the extended maximum, and the concept of confidence in assessments and level sets, we thus see that fuzzy decision analysis as presented in [24], [6], and [7] is an extension of standard Bayesian decision

theory, with several useful properties and allowing vaguer inputs. We thus feel that it should not be viewed as a replacement for decision analysis, but as an additional tool which may be of value. In the next section we examine the potential of using this theory to explain the observation of inconsistencies in probability assessments, and show how the extended maximum can be used to gain some insight into the "value of coherence."

#### 4.0 INCONSISTENCY AND INCOHERENCE

The problem is often encountered in applied decision analysis of DM's who provide "probabilities" which do not satisfy the probability axioms.

Although, typically, a decision analyst will point out the inconsistency to the DM, and ask the DM to try again, this is not entirely satisfactory. There is a great deal to be gained if the analyst can use a formal procedure to help the DM with this reconciliation. The analyst is, after all, hired to help the DM, and this tricky reconciliation is one aspect of the problem where the DM needs aid. Formal approaches to the problem have recently been developed [21], [9], [10], using traditional methods. Further discussion of the importance of the problem can also be found in [21] and [9]. In this section we give the basis of an approach using the fuzzy model.

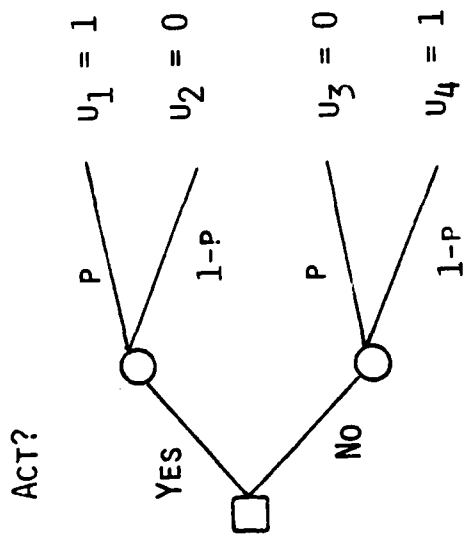
The essence of the approach is simple--if a DM is asked to provide just one number to describe his/her uncertainty, while implicitly modelling the uncertainty with a fuzzy probability, as in Figure 2(b), then with possibility equal to the membership of P in the fuzzy probability, the DM will respond with P. If the DM is asked in several different ways for the probability, it should come as no surprise that several different values of P are given. Furthermore, with the technical machinery of the previous section at our disposal, we need not ask for "reconciliation," since we can operate with the fuzzy values.

Within the fuzzy methodology, we are also able to model the concept of "perfect coherence" that is used in [21]. A perfectly coherent subject is one

who has integrated all of his/her knowledge about the world in a perfect fashion, and who would always give fully consistent responses to any questions concerning his/her probability assessments. Such a DM does not, of course, exist, but is a useful construct. Within the fuzzy methodology, it is necessary that a DM have only crisp probabilities, as in Figure 2(a), in order for such total coherence to be possible. Thus the "fuzziness" of a probability is an indication of the imperfection of the real DM. The ability to model that imperfection is a forte of the fuzzy methodology.

We define the "value of perfect coherence (VOPC)," in a manner analogous to the "value of perfect information (VOPI)" used in standard decision analyses. Within the fuzzy model, perfect information can be viewed as shifting our fuzzy probability to a crisp value of either zero or one. Our prior uncertainty of where that shift will be is, as in the non-fuzzy case, modelled by the original fuzzy probability. With this understanding, the VOPI concept can be easily extended to the fuzzy model. The effect of perfect coherence, on the other hand, may be viewed as shifting our fuzzy probability to some crisp value: the possibility that the shift will be to P being equal to the initial possibility of P in the fuzzy probability. The VOPC may then be calculated using a natural extension of the VOPI concept.

An example is shown in Figure 3. With the very simple decision tree of Figure 3(a), where the utilities are assumed crisp and either zero or one, and the probability being fuzzy as in Figure 3(b), the VOPI and the VOPC are shown superimposed in Figure 3(c). The values are, of course, fuzzy. It will be



(A)  
THE DECISION TREE

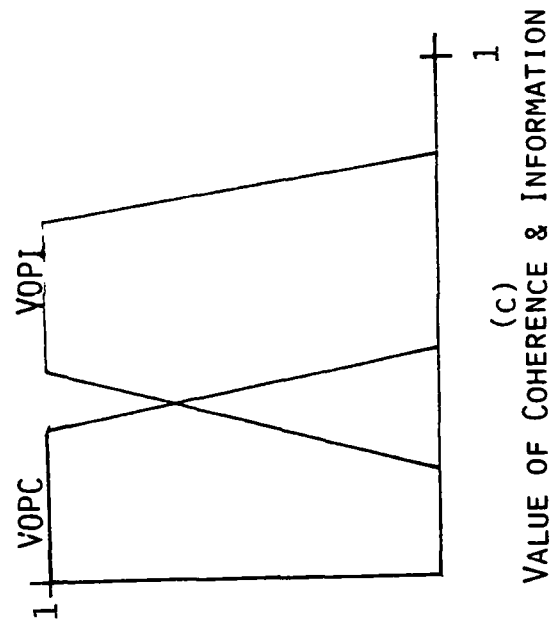
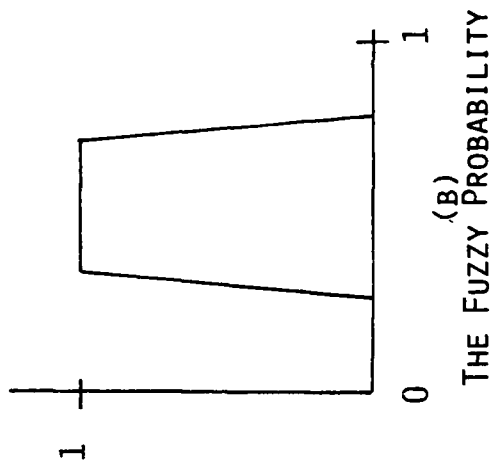


FIGURE 3

noted that using the partial order  $>$  defined in the previous section,

- a) the value of perfect coherence is greater than zero, and
- b) the value of perfect information is greater than the value of perfect coherence.

In [11] we develop the mathematics of fuzzy VOPI and VOPC, and show that for the tree of 3(a), the inequalities a) and b) always hold. These two results are both very intuitive, and give further encouragement for the use of the fuzzy model. We argue in [12] that the aim of a decision analysis is to improve the coherence of a DM, and thus that the value of coherence gives a method of calculating the value of an analysis prior to that analysis. Such a tool would be extremely useful, and we are currently pursuing these ideas with the aim of developing the technique further.

## 5.0 SUMMARY AND CONCLUSIONS

In this paper we have looked at two approaches that have been proposed using fuzzy set theory to help in the analysis of decision trees. We have concluded that the approach of Section 2.0 requires an operational definition of possibility in order to be sufficiently compelling to justify its use in preference to standard DA. We have suggested that this work should prompt us to consider the potential of using different connectives in different decision contexts. The max-min and the probability operators should be considered as extreme cases. The approach of Section 3.0 has been shown to be an extension of the standard paradigm, rather than contradictory as has sometimes been assumed. We have given an indication of how this extended theory may be used to help understand and deal with inconsistent probability (or utility) assessments, and have introduced the concept of the "value of perfect coherence."

We conclude with some further comments which are intended as pointers towards further research. We have looked only at the case of an individual DM, but, as pointed out in [8], there is an interpretation of the membership functions in the group decision-making context. The range of possible probabilities and utilities would arise from the differing values of the different DM's. Our theory provides a formal procedure for incorporating all these different opinions, but further work is needed to explore all the ramifications of the approach. We have not fuzzified the structuring process, but there is a strong potential for an application there--one could use the full power of



fuzzy sets and consider fuzzy events, rather than the crisp ones we have assumed, and use fuzzy measure theory to define the fuzzy probabilities. Another alternative would be to assume the decisions could be fuzzy. Whatever theories are developed, it should always be borne in mind that a very important aspect of providing a useful decision aid is providing an operational definition of the numbers used.

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